# On the Birkhoff–von Neumann decomposition


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# Context

An $n \times n$ matrix $\mathbf{A}$ is doubly stochastic if $a_{ij} \geq 0$ for all $i, j$ and $\mathbf{A}e = \mathbf{A}^T e = e$, where $e$ is the vector of all ones.

A doubly stochastic matrix has total support.

**Birkhoff's Theorem: $\mathbf{A}$ is a doubly stochastic matrix**

there exist $\alpha_1, \alpha_2, \ldots, \alpha_k \in (0, 1)$ with $\sum_{i=1}^{k} \alpha_i = 1$ and permutation matrices $\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_k$ such that:

$$\mathbf{A} = \alpha_1 \mathbf{P}_1 + \alpha_2 \mathbf{P}_2 + \cdots + \alpha_k \mathbf{P}_k \,.$$

- Also called Birkhoff-von Neumann (BvN) decomposition.
- Not unique, neither $k$, nor $\mathbf{P}_i$s in general.

# Combinatorial problem

INPUT: A doubly stochastic matrix $\mathbf{A}$.

OUTPUT: A Birkhoff-von Neumann decomposition of $\mathbf{A}$ as $\mathbf{A} = \alpha_1 \mathbf{P}_1 + \alpha_2 \mathbf{P}_2 + \cdots + \alpha_k \mathbf{P}_k$.

MEASURE: The number $k$ of permutation matrices in the decomposition.

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Exploit the BvN decomposition for preconditioning.

We show that the problem is NP-hard.

We also propose a heuristic and investigate some of its properties.

# Motivation

Consider solving $\alpha \mathbf{P} x = b$ for $x$ where $\mathbf{P}$ is a permutation matrix.

$$\alpha \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} \text{ yields } \begin{array}{l} x_4 = b_1/\alpha \\ x_3 = b_2/\alpha \\ x_1 = b_3/\alpha \\ x_2 = b_4/\alpha \end{array}$$

We just scale the input and write at unique (permuted) positions in the output. Should be very efficient (no dependency).

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Next consider solving $(\alpha_1 \mathbf{P}_1 + \alpha_2 \mathbf{P}_2)x = b$ for $x$.

# Motivation

Consider solving $(\alpha_1 \mathbf{P}_1 + \alpha_2 \mathbf{P}_2)x = b$ for $x$.

**Matrix splitting and stationary iterations**

for an invertible $\mathbf{A} = \mathbf{M} - \mathbf{N}$ with invertible $\mathbf{M}$

$$x^{i+1} = \mathbf{H}x^i + c, \quad \text{where} \quad \mathbf{H} = \mathbf{M}^{-1}\mathbf{N} \quad \text{and} \quad c = \mathbf{M}^{-1}b$$

where $i = 0, 1, \ldots$ and $x^0$ is arbitrary.

- Computation: At every step, multiply with $\mathbf{N}$ and solve with $\mathbf{M}$ (or apply $\mathbf{M}^{-1}$).
- Converges to the solution of $\mathbf{A}x = b$ for any $x^0$ iff $\rho(\mathbf{H}) < 1$.

# Motivation

**Theorem**

*Let* $\mathbf{A} = \alpha_1 \mathbf{P}_1 + \alpha_2 \mathbf{P}_2$ *and* $\alpha_1 \geq \alpha_2$*. Then,* $\mathbf{A}$ *is invertible if*

(i) $\alpha_1 \neq \alpha_2$,

(ii) $\alpha_1 = \alpha_2$ *and all connected components of the bipartite graph of* $\mathbf{A}$ *have an odd number of rows (and columns). If any such block is of even order,* $\mathbf{A}$ *is singular.*

Define the splitting $\mathbf{A} = \alpha_1 \mathbf{P}_1 - (-\alpha_2 \mathbf{P}_2)$.

The iterations are convergent with the rate $\alpha_2/\alpha_1$ for $\alpha_1 > \alpha_2$.

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Next generalize to more than two permutation matrices.

# Motivation: Let's generalize

Let $\mathbf{A} = \alpha_1 \mathbf{P}_1 + \alpha_2 \mathbf{P}_2 + \cdots + \alpha_k \mathbf{P}_k$ be a BvN decomposition of $\mathbf{A}$.

Assume $\alpha_1 \geq \cdots \geq \alpha_k$. Pick an integer $r$ between 1 and $k-1$ and split $\mathbf{A}$ as $\mathbf{A} = \mathbf{M} - \mathbf{N}$ where

$$\mathbf{M} = \alpha_1 \mathbf{P}_1 + \cdots + \alpha_r \mathbf{P}_r, \quad \mathbf{N} = -\alpha_{r+1} \mathbf{P}_{r+1} - \cdots - \alpha_k \mathbf{P}_k.$$

($\mathbf{M}$ and $-\mathbf{N}$ are doubly substochastic matrices.)

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Computation: At every step

- multiply with $\mathbf{N}$ ($k - r$ parallel steps).
- apply $\mathbf{M}^{-1}$ (or solves with the doubly stochastic matrix $\frac{1}{\sum_{i=1}^{r} \alpha_i}\mathbf{M}$); a recursive solver.

# Motivation: Let's generalize

Splitting $\mathbf{A} = \mathbf{M} - \mathbf{N}$ where

$$\mathbf{M} = \alpha_1 \mathbf{P}_1 + \cdots + \alpha_r \mathbf{P}_r, \quad \mathbf{N} = -\alpha_{r+1}\mathbf{P}_{r+1} - \cdots - \alpha_k \mathbf{P}_k.$$

**Theorem**

*A sufficient condition for $\mathbf{M} = \sum_{i=1}^{r} \alpha_i \mathbf{P}_i$ to be invertible: $\alpha_1$ is greater than the sum of the remaining ones.*

**Theorem**

*Suppose that $\alpha_1$ is greater than the sum of all the other $\alpha_i$. Then $\rho(\mathbf{M}^{-1}\mathbf{N}) < 1$ and the stationary iterative method converges for all $x^0$ to the unique solution of $\mathbf{A}x = b$.*

This is a sufficient condition; . . .and it is rather restrictive in practice.☹

# Motivation: Let's generalize

Open question: Identify other, less restrictive conditions on the $\alpha_i$ (with $1 \leq i \leq r$) that will ensure convergence.

The natural idea (put the mass in **M**) did not always work: we have examples with $k = 3$, $r = 2$ and $\alpha_1 + \alpha_2 > \alpha_3$ where the splitting did not converge.

Or, renounce convergence and use **M** as a preconditioner for a Krylov subspace method like GMRES.

# Motivation: Let's generalize

**M** as a preconditioner for a Krylov subspace method like GMRES.

... need generalize to matrices with negative and positive entries.

FACT: Any nonnegative matrix **A** with total support can be scaled with two (unique) positive diagonal matrices **R** and **C** such that **RAC** is doubly stochastic [Sinkhorn & Knopp,'67 and Knight, Ruiz and U.,'14].

Let **A** be $n \times n$ with total support and positive and negative entries.

**B** $=$ abs(**A**) is nonnegative and **RBC** is doubly stochastic.

We can write **RBC** $= \sum \alpha_i \mathbf{P}_i$.

# Motivation: Let's generalize

$\mathbf{B} = \text{abs}(\mathbf{A})$ and $\mathbf{RBC} = \sum_{i=1}^{k} \alpha_i \mathbf{P}_i$.

$$\mathbf{RAC} = \sum_{i=1}^{k} \alpha_i \mathbf{Q}_i \,.$$

where $\mathbf{Q}_i = [q_{jk}^{(i)}]_{n\times n}$ is obtained from $\mathbf{P}_i = [p_{jk}^{(i)}]_{n\times n}$ as follows:

$$q_{jk}^{(i)} = \text{sgn}(a_{jk}) p_{jk}^{(i)} \,.$$

**Generalizing Birkhoff–von Neumann decomposition**

Any (real) matrix $\mathbf{A}$ with total support can be written as a convex combination of a set of signed, scaled permutation matrices.

Define $\mathbf{M}$ (for splitting or for defining the preconditioner) as before.

# Linear algebraic problem and combinatorial problem

Reduce the complexity of the solver by reducing the cost of applying $\mathbf{M}$.

Find a BvN decomposition with the smallest number of perm. matrices.

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INPUT: A doubly stochastic matrix $\mathbf{A}$.

OUTPUT: A Birkhoff-von Neumann decomposition of $\mathbf{A}$ as $\mathbf{A} = \alpha_1 \mathbf{P}_1 + \alpha_2 \mathbf{P}_2 + \cdots + \alpha_k \mathbf{P}_k$.

MEASURE: The number $k$ of permutation matrices in the decomposition.

This is NP-hard.

# Heuristics: Birkhoff-like heuristic

1: $\mathbf{A}^{(0)} = \mathbf{A}$
2: **for** $j = 1, \ldots$ **do**
3: find a permutation matrix $\mathbf{P}_j \subseteq \mathbf{A}^{(j-1)}$
4: the minimum element of $\mathbf{A}^{(j-1)}$ at the nonzero positions of $\mathbf{P}_j$ is $\alpha_j$
5: $\mathbf{A}^{(j)} \leftarrow \mathbf{A}^{(j-1)} - \alpha_j \mathbf{P}_j$

**Birkhoff's heuristic: Remove the smallest element**

Let $\mu$ be the smallest nonzero of $\mathbf{A}^{(j-1)}$.

A step 4, find a perfect matching $M$ in the graph of $\mathbf{A}^{(j-1)}$ containing $\mu$.

# Heuristics: generalized Birkhoff heuristic

1: $\mathbf{A}^{(0)} = \mathbf{A}$
2: **for** $j = 1, \ldots$ **do**
3: find a permutation matrix $\mathbf{P}_j \subseteq \mathbf{A}^{(j-1)}$
4: the minimum element of $\mathbf{A}^{(j-1)}$ at the nonzero positions of $\mathbf{P}_j$ is $\alpha_j$
5: $\mathbf{A}^{(j)} \leftarrow \mathbf{A}^{(j-1)} - \alpha_j \mathbf{P}_j$

**Proposed greedy heuristic: Get the maximum $\alpha_j$ at every step**

At step 4, among all perfect matchings in $\mathbf{A}^{(j-1)}$ find one whose minimum element is the maximum.

Bottleneck matching: efficient implementations exist [Duff & Koster,'01].

# Experiments (heuristics)

$\tau$: the number of nonzeros in a matrix.
$d_{\max}$: the maximum number of nonzeros in a row or a column.

| matrix | $n$ | $\tau$ | $d_{\max}$ | Birkhoff | | greedy | |
|---|---|---|---|---|---|---|---|
| | | | | $\sum_{i=1}^{k} \alpha_i$ | $k$ | $\sum_{i=1}^{k} \alpha_i$ | $k$ |
| aft01 | 8205 | 125567 | 21 | 0.16 | 2000 | 1.00 | 120 |
| bcspwr10 | 5300 | 21842 | 14 | 0.38 | 2000 | 1.00 | 63 |
| EX6 | 6545 | 295680 | 48 | 0.03 | 2000 | 1.00 | 226 |
| flowmeter0 | 9669 | 67391 | 11 | 0.51 | 2000 | 1.00 | 58 |
| fxm3_6 | 5026 | 94026 | 129 | 0.13 | 2000 | 1.00 | 383 |
| g3rmt3m3 | 5357 | 207695 | 48 | 0.05 | 2000 | 1.00 | 223 |
| mplate | 5962 | 142190 | 36 | 0.03 | 2000 | 1.00 | 153 |
| n3c6-b7 | 6435 | 51480 | 8 | 1.00 | 8 | 1.00 | 8 |
| olm5000 | 5000 | 19996 | 6 | 0.75 | 283 | 1.00 | 14 |
| s2rmq4m1 | 5489 | 263351 | 54 | 0.00 | 2000 | 1.00 | 208 |

[At most 2000 permutation matrices, or accumulated a sum of at least 0.9999.]

# Experiments (linear systems)

Number of GMRES iterations

| matrix | ilu(0) | **M** with different *r* | | | |
|---|---|---|---|---|---|
| | | 1 | 2 | 16 | 32 |
| bp_200 | 2 | 38 | 31 | 22 | 23 |
| gemat11 | 168 | 1916 | 1606 | 620 | 297 |
| gemat12 | 254 | 2574 | 1275 | 570 | 386 |
| lns3937 | 348 | 1702 | 801 | 48 | 36 |
| mahindas | 37 | 225 | 158 | 43 | 32 |
| orani678 | 23 | 172 | 140 | 71 | 58 |

num. perm ($k$)

| matrix | $k$ |
|---|---|
| bp_200 | 5 |
| gemat11 | 48 |
| gemat12 | 34 |
| lns3937 | 57 |
| mahindas | 154 |
| orani678 | 1053 |

Number of nonzeros in **M**

| matrix | ilu(0) | 1 | 2 | 16 | 32 |
|---|---|---|---|---|---|
| bp_200 | 125 | 0.32 | 0.52 | 0.75 | 0.75 |
| gemat11 | 31425 | 0.15 | 0.21 | 0.69 | 0.88 |
| gemat12 | 31184 | 0.15 | 0.20 | 0.64 | 0.79 |
| lns3937 | 24002 | 0.15 | 0.25 | 0.59 | 0.65 |
| mahindas | 4744 | 0.12 | 0.16 | 0.29 | 0.36 |
| orani678 | 47823 | 0.04 | 0.04 | 0.05 | 0.06 |

# Concluding remarks

Open problem 1: Can we find less restrictive conditions for having a convergent solution?

Open problem 2: A better heuristic than the proposed greedy one? Approximation guarantee?

Open problem 3: Special, interesting cases that we can solve?

Open problem 4: Efficient algorithms for BvN decomposition?

# Thanks!

Thanks for your attention.

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## More information


- R. A. Brualdi, Notes on the Birkhoff algorithm for doubly stochastic matrices, Canadian Mathematical Bulletin, 25 (1982), pp. 191–199.
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# Known results for the num. of permutation matrices: Upper bound

*Marcus–Ree Theorem* ['59]: $k \leq n^2 - 2n + 2$ for dense matrices;

[Brualdi& Gibson,'77] and [Brualdi,'82]: for a sparse matrix with $\tau$ nonzeros $k \leq \tau - 2n + \ell + 1$ (containing $\ell$ submatrices with total total support; take $\ell = 2$)

# Known results for the num. of permutation matrices: lower bound

A set $U$ of positions of the nonzeros of $\mathbf{A}$ is called strongly stable [Brualdi,'79]: if for each permutation matrix $\mathbf{P} \subseteq \mathbf{A}$, $p_{kl} = 1$ for at most one pair $(k, l) \in U$.

**Lemma (Brualdi,'82)**

*Let* $\mathbf{A}$ *be a doubly stochastic matrix. Then, in a BvN decomposition of* $\mathbf{A}$*, there are at least* $\gamma(\mathbf{A})$ *permutation matrices, where* $\gamma(\mathbf{A})$ *is the maximum cardinality of a strongly stable set of positions of* $\mathbf{A}$*.*

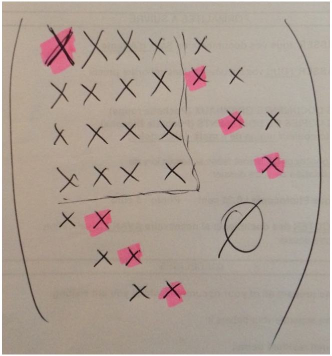
$\gamma(\mathbf{A}) \geq$ the maximum number of nonzeros in a row or a column of $\mathbf{A}$

[Brualdi,'82] shows that for any integer $t$ with $1 \leq t \leq \lceil n/2 \rceil \lceil (n+1)/2 \rceil$, there exists an $n \times n$ doubly stochastic matrix $\mathbf{A}$ such that $\gamma(\mathbf{A}) = t$.

# Known results for the num. of permutation matrices: lower bound

$\gamma(\mathbf{A}) \geq$ the maximum number of nonzeros in a row or a column of $\mathbf{A}$

[Brualdi,'82] shows that for any integer $t$ with $1 \leq t \leq \lceil n/2 \rceil \lceil (n+1)/2 \rceil$, there exists an $n \times n$ doubly stochastic matrix $\mathbf{A}$ such that $\gamma(\mathbf{A}) = t$.



# Combinatorial problem: Sketch of the proof of NP-completeness of the decision version

Reduction from 3-PARTITION: given an array $A = \{a_1, \dots, a_{3m}\}$ of $3m$ positive integers, a positive integer $B$ such that $\sum_{i=1}^{3m} a_i = mB$ and $B/4 < a_i < B/2$, does there exist a partition of $A$ into $m$ disjoint arrays $S_1, \dots, S_m$ such that each $S_i$ has three elements whose sum is $B$.

$$\mathbf{T} = \begin{bmatrix} \frac{1}{m}\mathbf{E}_m & O \\ O & \frac{1}{mB}\mathbf{C} \end{bmatrix} \text{ where } \mathbf{C} = \begin{bmatrix} a_1 & a_2 & \dots & a_{3m} \\ a_{3m} & a_1 & \dots & a_{3m-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_2 & a_3 & \dots & a_1 \end{bmatrix}.$$

$\mathbf{E}_m$ is an $m \times m$ matrix of 1s; $\mathbf{T}$ is doubly stochastic.

$\mathbf{T}$ has a decomposition with $3m$ permutation matrices iff we can solve the 3-PARTITION instance $\{a_1, \dots, a_{3m}\}$.

# Heuristics (properties)

Let $\mathbf{S}$ be $n \times n$ doubly stochastic, $m = n!$.

The map $f : (c_1, \ldots, c_m) \to c_1 P_1 + \cdots + c_m P_m$ from the $(m-1)$-dimensional simplex $\{(c_1, \ldots, c_m) : c_i \geq 0, \sum_i c_i = 1\}$ to $\Omega_n$ is a continuous surjection, and for each $\mathbf{S} \in \Omega_n$, $f^{-1}(\mathbf{S})$ is a compact convex set.

**Lemma**

*There is an extreme point of $\mathbf{S}$ with the minimum number of permutation matrices.*

**Lemma**

*A heuristic of the generalized Birkhoff family finds an extreme point of the convex polytope $f^{-1}(\mathbf{S})$.*